The Aftermath

Leon Horsten and Philip Welch

To improve efficiency and to achieve faster throughput, Prejudgement Reception outside the Pearly Gates has been reorganised, and it has been decided that philosophers, scientists, mathematicians, theologians, and literary theorists would be processed centrally in batches from now on, and indeed only in infinite batches.

Whilst waiting in the "Prejudgement Antechamber" (more commonly known locally as the Limbo Room) for the requisite numbers to turn up, the transmigrating thinkers from various worlds discover their new-found powers of perception, which include being able to take in and process infinite amounts of information at a glance: the mathematicians amuse themselves by playing two-person infinite games of perfect information, they nod knowingly at the confirmation of Goldbach's Conjecture, and they discover the consistency (or otherwise) of their favourite theories, such as Peano Arithmetic or Zermelo-Fraenkel set theory; there are groans of dismay and even some emotional moments when they learn that the first zeroes of the Riemann zeta function off the critical line occur at distances of the order of $10^{10^{47}}$; the cosmologists have mixed feelings about the partial confirmation of the Big-Bang theory; the one and only absolute reading of Genesis dismays the literary theorists, whilst the biologists are crestfallen at the triumph of Creationism.

When a countably infinite number have been gathered in, they are told that Judgement will proceed: Enter God (looking entirely like you would expect, as from a Blake watercolour); He makes the following announcement: "Welcome to the Judgement Precinct. You will shortly be proceeding next door to the Hall of Judgement, where the Final Reckoning will take place. As you well know, those of Sufficient Virtue will gain admittance within these Gates, but those of you deemed of, how shall We put it? Of *Insufficient* Virtue will have to,…" here He trailed off, looking tired and grave for a moment, "…will have to go to…, go to the… Other Place."

Then, brightening up, He said that, as was widely rumoured, He was merciful. Each would have a second opportunity of Redemption by meditating on His, God's, Judgement. There were, He continued, some differing courses of action open to them: The first such was that they could use their own, now enhanced, powers of judgement (both of morals and of other relevant factors of esteem), and of searching through the souls of others (which they had not noticed whilst all the fun with Goldbach's Conjecture, etc., was going on).

They might then care, the Almighty continued, whilst in the Hall of Judgement, to introspect and also to come to some judgement about themselves and accordingly decide what their own fate should be. "However," said God, "it is important for you to understand that your moral judgement about yourself or others will mostly agree with mine. Indeed each individual, if he or she were to judge all those present, would only make finitely many mistakes. However, I make no particular promise as to how you might judge yourselves, other than that you need not fear misjudging the beam in your own eye, compared to the mote of others: I can assure you, you will be as adept, and as fair, at self-judgement as at adjudicating other souls."

God gave his solemn promise on this. This they all believed (they were generally in a chastened mood after the debacle over Evolution).

"You see, getting it right, even if it is a self-judgement of Insufficient …" The Almighty coughed, somewhat embarrassed, and started again: "I regard correct moral self-judgement, even in the case of Insufficient…, as pertaining to some form of repentance, *final value-added* confession so-to-speak, worthy of redemption.

"Unfortunately, if infinitely many of you make bad judgements and assess your own Judgements incorrectly, then, in the interests of… then I am afraid you will *all*… you will all have to go down… the Other Place."

Of course, He continued, they might try exercising their inalienable right to Free Will (at this point the Lord faintly smiled): some other plan or tactic could be deployed. They would even be allowed to choose to appoint one person to represent the whole group, a champion, so to speak: if the appointed person arrives at a correct self-judgement, then everyone is saved, if not, all are sent down. What would not be permitted (on pain, or even – they were left in no doubt – infinite pain, which would be the inevitable outcome of collective and total expulsion into the hands of You Know Who), would be for them to start now informing each other of their

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moral judgements on each other, or even their own self-judgements, thus pre-empting the One and Final Judgement. Discussion now would be strictly limited to whether to use their own judgements or which action or tactic to take.

After they had decided on a course of action, they would be ushered in to the Judgement Hall, where, when the Reckoning was completed, they would have the opportunity to observe His, God's, Judgements: each person would find that he or she had a hat on, either white (indicating admission), or else a somewhat orangey red. Each person would then be required to indicate his or her final choice of the quality of his own Virtue on the personal Self-Assessment Form (VG51) (Please tick Box 1 *Sufficient* or 2 *Insufficient Virtue* [Required Boxes – black ink only]). If the whole group would opt to be represented by a single chosen individual, then the procedure would be simpler: only the representative would enter the Judgement Hall, and the others would wait outside.

God then gave them 10 minutes to come to a collective decision.

Paradise Regained?

After short silence then
And summons read, the great consult began

Milton - *Paradise Lost* Bk.I

After God left the room, there was a brief moment of stunned silence, and then uproar. Turmoil, shouting, and even (one would think, for such a place) unseemly threats. The normally sober Prejudgement Area was transformed into a scene more worthy of Pandemonium. Eventually a Mathematician held the floor.

The Mathematician's Argument
(The Shade of Hermann Kahn)

The Mathematician spoke thus: "We now have powers to absorb and retain infinite amounts of information, I propose we invoke the Axiom of Choice (*AC*). We can adopt the following strategy: we give ourselves numbers $0, 1, 2, \ldots, k, \ldots$ (since I perceive we are denumerably many). Each person remembers his or her number. Now consider all possible listings of judgements of Sufficient and Insufficient Virtue, such as the list J: $j_0 = I(nsufficient)$, $j_1 = S(ufficient)$, $j_2 = I$, $j_3 = S, \ldots, j_n = S, \ldots$, with, for example, j_3 being the outcome on person number 3. There are infinitely many such lists of I's and S's, but one of them is God's Judgement list. Let us say two possible judgement listings are *equivalent* if from some point k on, the j_k in the two listings are identical. This is an equivalence relation. Let us now pick, and so collectively agree on, for each equivalence class E of equivalent listings under this relation, a single representative list from this class, call it J_E, say." Seeing a look of puzzlement on some of the faces of the deconstructionists, he leaned towards the foremost and said, "This is a kind of proxy for the class E – it does not matter *which* J_E from E it is, as long as we all agree on it. Then at the One and Final Judgement, we use the hat colours we can see and our numbering of ourselves to make the list of God's Judgements J_G. The point is that standing in our enumerated order and looking forwards, we shall all see in any case the same J_G (apart from God's judgement on ourselves and those behind us with earlier numbers); hence we all know in which equivalence class it lies, let us call it E_G; and we all have agreed on our response list for this class: J_{E_G}. When we have to commit to a self-judgement, person numbered k will write down the value of the kth element of the listing J_{E_G} that we have all previously agreed on. The list of our responses may not be exactly God's listing J_G, but, and this is the point, it only differs at most on finitely many people at the start of the sequence.

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Hence only finitely many of us will be incorrect at most, and only these will be punished. We should agree on this plan, which results in a possible sacrifice of, at worst, finitely few."

Suddenly seeing Pascal in the crowd, he called out, "See that man over there? He is a mathematician like me: He will agree with me!"

There was a murmuring and a shifting of feet, whilst the implications of this strategy were absorbed. Some even started backing away from the Mathematician in the fear of being assigned a low number.

The man pointed out by the Mathematician elbowed his way to the front, shouting all the while: "No! No! Not at all! Please, please, listen. This is quite wrong" (but in French).

The uproar slowly calmed, and a whispering broke out amongst the crowd.

The Shade of Pascal's Argument (for it is indeed he):

(Pascal pauses for quiet.)

"The previous speaker's argument is specious and faulty. Who believes in this so-called Axiom of Choice? It is a chimæra. Suppose it would be true. Then the argument just given can be used to justify almost anything. For example, suppose God says to us 'I have chosen a function $F : \mathbb{R} \to \mathbb{R}$' (rather than a judgement sequence J_G). Suppose God allows each of us to choose, instead of Sufficient or Insufficient, a real number. For each real number x, He then shows those who chose x the graph of F without the point $(x, F(x))$ and allows each to 'guess' the value $F(x)$ to gain his/her salvation. Of course we cannot do it! The probability of finding the correct $F(x)$ is that of choosing the right real number out of all the uncountably many real numbers that exist (as Herr Cantor has shown). But the Mathematician here would have you believe that we could pick, by this so-called 'Axiom' of Choice, a representative function of each equivalence class of functions from $\mathbb{R}$ to $\mathbb{R}$, which only differ on finitely many x; he would then say that our representative function can only differ from God's function at finitely many places. Hence the conclusion is that, given God's choice of function F, the probability would even be certainty that any one of us would be correct! We could only be wrong finitely often! This is nonsense, and you should not listen to it.

"The situation is laughably simple. God has promised us, and which of you…", he now glowered at the biologists, "…which of you can now doubt His Word? He has promised us that our moral judgements about others will coincide with His when we enter the Hall of Judgement; we shall all have the evidence, even an infinite amount of evidence, that He is wise, just, and, more to the point, He agrees with us. It is true that we have not been vouchsafed that we are correct in our own self-worth. Truly, in order to enjoy His Gift of full Redemption, God in His Mercy has allowed us one more shot at it. Listen to your own moral reason."

(Having gained their full attention, Pascal pauses once more for breath; all the while he is beaming at his audience. He is, after all, feeling lucky.)

"We should observe the following: although God says that we *may* be wrong about our self-judgements, as we could make finitely many errors in our judgements, what is the probability that we are wrong about ourselves? We shall even have an abundance of evidence in the Hall of Judgement that we are good, nay even *almost perfect*, judges of peoples' souls! Given that each and every one of us will have been given an infinite amount of evidence that our faculties of moral judgement are in accord with that of the Supreme Being, the probability for each one of us that he is wrong about his self-judgement is surely one of Mr Newton's infinitesimals? We should choose one worthy representative, perhaps Cardinal Richelieu here, and let our lot depend on the correctness of his own moral self-judgement. Listen to your own God-given Faculty of Judgement, and we shall all be saved."

The uproar returned, agitated with especial vigour by the Calvinists.

What should they do? Is Pascal correct?

Someone shouts, they have performed the impossible and have demonstrated that the Banach-Tarski paradox is no paradox at all, but that the Axiom of Choice is true. Someone holds up a sphere and with some *legerdemain* reassembles it into two balls of the same size. Others try it, succeeding in making three, four, more balls, sometimes of even greater size than the original. There are cries of astonishment and joy.

They come to a rapid decision. Despite Pascal's shouts of "*Casuistique…*", they put the Mathematician's strategy into effect. Pascal in his rage kicks one of the spheres, that now spontaneously falls apart and even comes together again as a duplicate pair! They choose numbers for themselves, and, invoking the Axiom of Choice, representatives J_E for each equivalence class E of finitely differing sequences of Judgements of White and (Orangey-)Red.

They then file in their numerical order into the Hall of Judgement, and they all find, *mirabile dictu*, that they have hats on their heads. They form an orderly line and, each looking ahead at the later numbers, see that indeed his or her judgement has agreed with the Almighty's. He has been as good as His Word. It looks as if the Mathematician's strategy is going to work.

However … Machiavelli is number 472 in line. He is unhappy. Looking down the line, the resulting agreed-upon judgement list J_{E_G} would tell him to give a different response than what he would give if were he to use his own self-judgement.

He is thinking of defecting.

Promises, Promises

> The promise given was a necessity of the past:
> the word broken is a necessity of the present.
>
> after Machiavelli – *The Prince*

The Shade of Machiavelli's Argument (sotto voce):

Whilst looking up the line, he sees that, in fact, using his own judgement, he has made no mistakes at all there: it agrees completely with God's Judgement as revealed by the hats on the people.

He surreptitiously sneaks a look backwards along the line at the people behind him. This only serves to confirm his belief in his self-judgement and that he should defect from the

Mathematician's strategy: since again for these, albeit now finitely many, cases, it turns out that his judgement has also accorded with the Almighty's.

He reasons thus: " The Mathematician's strategy may work, but if the response from His strategy that I enter for my self-judgement is wrong, I am doomed. It could be that infinitely many of the people in front of me are in the same situation as I am: They are all in line looking forwards at those with later numbers, and they have a self-judgement that disagrees with the Mathematician's strategy. If so, then many are in effect disagreeing with God's judgement, but, as far as they can see, perhaps only on themselves. In that case, if infinitely many of them are reasoning in the same way, and then defect and write down their contrary self-judgement, we are all doomed. However they may not. The evidence is that my judgement on all these people is perfect. God has assured us that on balance we are no worse at judging ourselves than others. So although I may still make a single error, namely, that of my self-judgement, my judgement is not worse than what was given by this Axiom of Choice? And, anyway, what does the Axiom give us but a randomly chosen representative of the equivalence class? So I should defect and use my judgement irrespective of what those in front of me do."

Is Machiavelli's reasoning correct? What should he do?

Optional Endnote – in lieu of a Discussion

The relevant facts of the predicament in which the Dead Souls find themselves can be summarised as follows:

(i) God has a judgement *S*(ufficient) or *I*(nsufficient) on everyone present;

(ii) Anyone may also draw up a private judgement on any or all of the others (as to *S* or *I*), including on himself or herself;

(iii) Each person's assessment is highly reliable: If he or she were to assess everyone in the countable set, he or she would only make finitely many errors (but one of these errors may be his or her own self-assessment, however the latter will be neither more nor less reliable than his or her judgement of others);

(iv) This reliability will become apparent when everyone is able to see God's assessment of every higher-numbered person (thus not including himself or herself);

(v) *Either* everyone must then record and return his own self-assessment, *or* the group must collectively nominate in advance a single champion who will record and return her own self-assessment as proxy for them all.

The outcomes are then as follows:

(1) If they nominate a single "champion," and she gets her self-assessment wrong, all are damned; if she gets her self-assessment right, all are saved (regardless of God's assessment of *S* / *I* of them).

(2) If they opt to return self-assessments, and infinitely many make an error, then all are damned (*tout court*, so again regardless of God's assessment of them); if there are only finitely many errors, then those making correct self-assessments are saved.

Intuitively, the dilemma that the group is facing can be described as follows. One course of action (Kahn's ploy) will with mathematical necessity result in only finitely many people being "sent down," but with no *a priori* bound on that finite number. Another course of action (appointing a champion) possibly results in infinitely many people sent down, but this outcome seems "infinitely unlikely." Deciding which course of action is best calls for a deeper analysis of the notion of likelihood that is involved. (That they can only see people later in line with higher numbers is in fact a red herring. Kahn unnecessarily imposes this: It would suffice were each person to see all the hats except his or her own; the numbering of people is needed here to impose an ordering.)

Mathematicians have for a long time known that there is a discrepancy between our intuitive notions of probability and the concept of probability measure; as often as not, these are mediated through a use of the Axiom of Choice. Chris Freiling in [Freiling 1986] provides an argument that if we associate to every real number x a countable set of real numbers $C(x)$, then if we throw two darts at the real line, hitting at, say, x and y, then intuitively, there must exist x, y with $x \notin C(y)$ and $y \notin C(x)$; indeed there must exist many such x, y pairs. However, an easy argument (due to Sierpiński) shows that the assertion that for every such C, there do exist x, y with $x \notin C(y)$ and $y \notin C(x)$, is equivalent to the negation of Cantor's Continuum Hypothesis. In other words, it shows that there is no well-ordering of the real continuum of order type $\aleph_1$.

This can then be extended, as Freiling notes, to an argument against the well-ordering of the real continuum irrespective of its order type. The intuitive notion of probability that is being used – that one defines probabilities by limit ratios of hits and misses through thrown darts – leads to a countably additive translation-invariant measure, which contradicts *AC*. That there are paradoxical effects of *AC* using "nonmeasurable" sets is, of course, also well known through the earlier Banach-Tarski paradox that heavily uses *AC* to decompose a billiard ball using nonmeasurable sets and reassembles them into two spheres each the volume of the earth.

The relation of *AC* to "hat problems," such as the story illustrates, is perhaps less known, and only in the last few years has there been any literature describing this.[1] Hardin and Taylor in [Hardin & Taylor 2008] and [Hardin & Taylor 2008a] present an accessible account.

The article throws a light on our notions of intuitive probability, and in particular on what criteria for decision-making or subjective probabilities can be sensibly carried over to the (admittedly theoretical) realm of the infinite. There is an honourable history of this, for example when Kreisel

[1]The second author heard the first scenario of the ω-sequence of hats and what became Kahn's argument in a conversation with Yuri Matiyasevich; and from this developed Pascal's argument on functions from $\mathbb{R}$ to $\mathbb{R}$ by simple generalisation. Hardin and Taylor credit Yuval Gabay and Michael O'Connor with the first puzzle, but their arguments are very generally stated and cover the second puzzle expounded by Pascal and others, too.

advocated hard for forms of "meta-recursion" on countable ordinals that he hoped would illuminate the various roles of the concept "finite" in ordinary computable-theoretic arguments using finite numbers.

It has been recognised that nonmeasurable sets present problems for rational decision theory (section 9.2.1,Binmore 2009). Moreover, this is precisely the situation that we are confronted with in the story. Consider Kahn's equivalence classes of judgement sequences against the background of the assumption that the Axiom of Choice indeed holds. Then each equivalence class must be a nonmeasurable set in Cantor space. So the probability that the actual sequence of judgements belongs to any given equivalence class is undefined. But such probabilities have to be defined in order to perform orthodox utility calculations for the whole group.

There are several possible ways out:

(1) One can deny the Axiom of Choice. Then one can consider models of Zermelo-Fraenkel set theory (without Choice), in which all sets in Cantor space are measurable [Solovay 1970]. This in itself would not solve *all* problems, for there still will be infinite ultilities that will have to be "tamed" somehow.
(2) One can accept the Axiom of Choice, as is done in the story. Then there are again several options:
    (a) One can employ "unorthodox" probability theories. One way of doing this is to somehow extract utilities from upper and lower probabilities (section 9.2.2, Binmore 2009). Another way of doing this is to appeal to techniques of nonstandard analysis. Ordinary nonstandard measure theory will not help here, for Cantor space will be an external object in this setting. But perhaps a clever combination of standard and nonstandard ideas will yield a natural way of calculating utilities ([Wenmackers & Horsten], [Benci, *et al.* 2011]).
    (b) One can argue that in scenarios such as that which is painted in the story, the canons of rationality do not determine a unique course of action.

The reader should note that the use of *AC* to establish the existence of a Kahn-like strategy is unavoidable: By results of Solovay and Shelah ([Shelah 1984]) it is consistent with the axioms of Zermelo-Fraenkel set theory (*ZF*) that every subset of the real line has the Baire property (that is, it has a symmetric difference with an open set that is meagre). However, from a strategy that ensures only finitely many incorrect guesses in the first puzzle, one can derive the existence of sets without this property. This implies that the existence of such a strategy cannot be proven in *ZF* alone: *AC* is required.

One should further note that Pascal is both right and wrong: Kahn could indeed deliver the argument that Pascal presents on functions from reals to reals, and he (Kahn) would be entirely correct. Pascal, however, then draws the wrong conclusion – that it is so absurd as to be unbelievable. Note that the argument shows that *given* a judgement function F then for a randomly given x, the strategy yields the correct expectation: $F(x)$ is correctly guessed with probability 1 (*i.e.*, on a set of measure 1). It would be the wrong conclusion to draw, given an x and then F that is chosen "somehow randomly," that the probability of being correct at x is still 1. However, note that Pascal does not say this: he makes the correct statement but still thinks it absurd. He simply does not believe in the Axiom of Choice. By the previously described arguments, Kahn would not be able to prove the existence of a strategy without it (both Shelah and Solovay being alive, and so not in the Prejudgement area, Pascal could not know this fact yet). However he then goes on to present a somewhat vague argument that an infinite (actually cofinite) amount of evidence one way yields up an infinitesimal probability of the contrary happening. That he proposes "therefore" the One-Champion solution, and suggests Richelieu as a "worthy" candidate, is intended as light irony: Even if Richelieu is unfairly maligned by history, Pascal as a committed Jansenist, and with his well-publicised views on casuistry, is unlikely to trust someone who has taken the vows of the Order of the Society of Jesus.

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