Russell and Fine on Variable Objects*

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Abstract

In this article we compare Fine's theory of arbitrary objects with the theory of variables that Russell formulated in his *Principles of Mathematics*. We argue that Russell's early theory of variables can be seen as a prefiguration of Fine's theory of arbitrary objects. The main difference between Russell's theory and Fine's account lies in their account of dependence relations between variables. Fine develops a stable view of dependence between arbitrary objects, whereas no such view is presented in *Principles of Mathematics*. We then sketch how Russell developed the notion of dependence between arbitrary objects in some of his subsequent works.

> *The notion of the variable is one of the most difficult with which Logic has to deal, and in the present work a satisfactory theory as to its nature, in spite of much discussion, will hardly be found.*
>
> Russell, *Principles of Mathematics*, pp. 5–6

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1 Introduction

In the first half of the 1980s, Kit Fine developed a theory of arbitrary objects [Fine 1983], [Fine 1985], [Fine 1985b]. Fine was not only interested in developing the 'pure' metaphysical theory of arbitrary objects. From the outset, he wanted to develop *applications* of arbitrary object theory to specific areas of philosophy. One of these areas concerns certain episodes and projects in the history of philosophy. Fine believes that arbitrary object theory might shed light on platonic theories of Form, for instance,[1] and that it can shed light on the debate between Frege and some mathematicians of his time on the existence of 'variable numbers'.

Fine's suggestions have until now not been pursued far. Indeed, over the past three decades arbitrary object theory has been largely (but not completely) dormant. In this article, we want to take a small step in pursuing one of Fine's suggestions. Specifically, our primary aim is to investigate Russell's account of variables in *The Principles of Mathematics* [*PoM*, 1903] through the lens of Kit Fine's arbitrary object theory. We will also attempt to indicate, albeit briefly, possible connections between Fine's theory of arbitrary objects and some views of functions that Russell developed after *PoM*, including the view that is presented in *Principia Mathematica* [*PM*, 1910–1913].

At first sight, our principal aim may appear to be an unpromising avenue. After all, Fine wrote that in *PoM*, Russell was following Frege's lead in rejecting the existence of arbitrary objects [Fine 1983, p. 55]:

> Given [Frege's] own theory of quantification, it was unnecessary to interpret the variables of mathematics as designating variable numbers; and given the absurdities in the notion of a variable number, it was also unwise. [. . .]
> Where Frege led, others have been glad to follow. Among the many subsequent philosophers who have spoken against arbitrary objects, we might mention Russell (PoM, pp. 90-91) [. . .]

But the situation is more complicated, . . . and more interesting.

It is unlikely that Russell knew of Frege's misgivings about the idea of arbitrary objects. Frege published a short article in a *Festschrift* for Boltzman in 1904 in which he inveighed against the mathematician Czuber's use of variable numbers [Frege 1904], but this appeared after *PoM* was written and published. There is an earlier text by Frege in which

[1] See [Fine 2017].

he attacks the use of variable numbers in the foundations of mathematics [Frege 1898/9].[2] But this is an unpublished manuscript, which Russell is unlikely to have had access to.

More importantly, as we will show in this article, Russell's account of variables in *PoM* has much in common with Fine's theory of arbitrary objects. Indeed, it is striking how Russell not only correctly perceives, at least in broad outlines, how a theory of arbitrary objects should be developed: he also anticipates theoretical difficulties that a satisfactory theory of arbitrary objects will have to overcome.

The structure of this article is as follows. In the next section, we give a concise overview of Fine's metaphysical theory of arbitrary objects. In section 3, we describe the background of Russell's account of variables in *PoM*. Subsequently, in section 4, we present a detailed account of Russell's theory of variables in *PoM*. We will see that there are tensions in Russell's views, and that he arguably did not arrive at a metaphysical view of variables that fully satisfied him. In section 5, we compare Russell's account of variables in *PoM* with Fine's theory of arbitrary objects, and observe that they to a significant extent overlap. In section 6, we close the paper by sketching possible connections between Fine's theory of arbitrary objects and theories of functions that Russell developed after *PoM*.

2 Fine on arbitrary objects

Fine begins his book on arbitrary objects as follows [Fine 1985, p. 6]:

> There is the following view. In addition to individual objects, there are arbitrary objects: in addition to individual numbers, arbitrary numbers; in addition to individual men, arbitrary men. With each arbitrary object is associated an appropriate range of individual objects, its values: with each arbitrary number, the range of individual numbers; with each arbitrary man, the range of individual men. An arbitrary object has those properties common to the individual objects in its range. So an arbitrary number is odd or even, an arbitrary man is mortal, since each individual number is odd or even, each individual man is mortal. On the other hand, an arbitrary number fails to be prime, an arbitrary man fails to be a philosopher, since some

[2] This text was dated by the translators to "1898/99 or later, probably not after 1903" [Frege 1898/9, p. 157].

> individual number is not prime, some individual man is not a philosopher.
>
> Such a view used to be quite common, but has now fallen into complete disrepute.

Yet this is, in broad outlines, the view that Fine develops in his book. So it is worth describing Fine's theory of arbitrary objects in some more detail.

The core thesis of arbitrary object theory is that beside specific objects, there are arbitrary objects. Beside all specific men, for instance, there exists the arbitrary man.

Arbitrary objects metaphysically differ from specific objects. With any arbitrary object, a *value range* is associated. There is a sense in which an arbitrary object can take specific objects as its *value*. The man in the street could be John Rogers. It could also be James Goodall, and so on, for every specific man.

It would not be very wrong to say that an arbitrary object has exactly those properties that all specific objects in its value range share. For instance, the arbitrary man is a rational being. Fine calls this principle the *principle of generic attribution* [Fine 1983, p. 59]. But this principle has to be qualified. For instance, all specific men are specific objects, but the arbitrary man is not a specific object. This gives rise to Fine's distinction between *generic* and *classical* readings of conditions [Fine 1983, p. 63–65]. We need not go into this somewhat subtle aspect of Fine's theory here. It suffices for our purposes to recognise that 'x is an F' in the classical reading of the condition or predicate 'F' entails 'x is an individual F', while it does not in the generic reading of 'F'. The principle of generic attribution is concerned only with conditions in the generic reading.

A cornerstone of Fine's theory of arbitrary objects is the relation of *dependence* between arbitrary objects. Fine distinguishes between *independent* arbitrary objects and *dependent* arbitrary objects. With each value-range of specific objects F's, *one and only one* independent arbitrary F is associated [Fine 1983, p. 71]. In addition, there are many *dependent* arbitrary F's: their dependence on the independent F is witnessed by the fact that the value that a dependent F takes depends on which value the independent F takes. And then there are F's that depend on dependent arbitrary F's, and so on. Let us turn again to our example of arbitrary men to make this more concrete. The arbitrary man is an independent arbitrary object: call this person a. Then there also is the father of a. This is a dependent object. If a takes John Rogers as its value, then the father of a takes Tom Rogers (John's father) as its value. If a takes James Goodall as its value, then the father of a takes Tobias Goodall as its value. The father of a is a dependent object *of rank 1*. But we can go on to consider the father of the father of a, which is a dependent object of rank 2, and so on.

Concerning independent arbitrary objects, Fine states that an independent arbitrary object x is numerically identical to an independent arbitrary y if the value range of x is the same as the value range of y [Fine 1983, p. 69]. But he observes that this gives rise to an apparent difficulty [Fine 1983, p. 70]:

> Although [the general theory of arbitrary objects] does not require it, we will want to make an application to the variable-signs of mathematics. In a sentence such as 'Let x and y be two arbitrary reals', we will want to say that the symbols 'x' and 'y' refer to two unrestricted and independent arbitrary reals. But to which? It is natural to suppose that 'x' and 'y' refer to two unrestricted and independent arbitrary reals. But [by my criterion of identity for independent arbitrary objects] there is only one such real. So do either of the symbols refer to it and, if so, to what does the other refer?

Fine proposes that we solve this problem in the following way [Fine 1983, pp. 70–71]:

> Perhaps the most natural [solution to this problem] is one that makes x and y not be independent at all. There is a unique arbitrary pair of reals, p; it is the independent arbitrary object whose values are all the pairs of reals. We may then take x and y to be the first and second components of this arbitrary pair. More exactly: x and y will each depend on p and upon p alone: when p takes the value (i, j), then and only then will x take the value i and y take the value j.

But this does not appear to be *the most natural thing to say*.[3] The most natural solution is to take appearances to be correct, i.e., to take x and y to be *two* independent variables.[4] This means giving up Fine's criterion of identity for independent arbitrary objects. In later work, Fine refrains from claiming absolute validity for the criterion of identity for independent arbitrary objects that he proposed in [Fine 1983].

There is also *higher-order* arbitrariness. For instance, consider the value-range F' consisting of all arbitrary men that have a class of specific men as their value-range. There will be one independent arbitrary man with

[3]This point is made in [MacNamara 1988, p. 306].

[4]This line is developed in [Horsten 2019, section 9.4].

value-range F', i.e., one independent 'arbitrary arbitrary man', and there will be many dependent arbitrary arbitrary men. In this way, a hierarchy of arbitrariness unfolds: there is first level arbitrariness, there is second level arbitrariness, and so on. Fine argues that some sort of stratification is required to stave off Russell-like paradoxes.

This, in a nutshell, is Fine's metaphysical account of arbitrary objects. Somewhat surprisingly, when pushed on the question of realism, Fine expresses reservations [Fine 1985, p. 7]:

> If now I am asked whether there are arbitrary objects, I will answer according to the intended use of 'there are'. If it is the ontological significant sense, then I am happy to agree with my opponent and say 'no'. I have a sufficiently robust sense of reality not to want to people my world with arbitrary numbers and arbitrary men. Indeed, I may be sufficiently robust not even to want individual numbers or individual men in my world. But if the intended sense is ontologically neutral, then my answer is a decided 'yes'. I have, it seems to me, as much reason to affirm that there are arbitrary numbers in this sense as the nominalist has to affirm that there are numbers.

Fine's theory of arbitrary objects is thus meant to be neutral as to its ontological implication. Yet, if we read his theory in a 'naive' spirit, we can readily see its similarity to Russell's view of 'variables' in *PoM*, which we shall discuss below.

3 The background of Russell's theories of variables in *PoM*

In *PoM* Russell develops a logicist position: he attempts to define all the relevant notions of pure mathematics in terms of logical notions and to deduce the theorems of mathematics from these definitions only using logical inferences. *PoM* does not only outline the whole project; it also contains his philosophical account of key notions such as *proposition*, *class*, *propositional function*, and *variable*. In this section, we introduce the metaphysical view presented in *PoM*, in the context of which he develops his account of these concepts.

In *PoM* Russell introduces a very generous ontology. On his view, '[w]hatever may be an object of thought, or may occur in any true or false proposition, or can be counted as *one*' is called a *term*, where the word 'term' is taken to be synonymous with 'unit', 'individual', and 'entity' [*PoM*, p. 53]. Indeed, everything we can talk about is considered to be a term: '*Being* is that which belongs to every conceivable term, to every possible object of thought—in short to everything that can possibly occur in any proposition, true or false, and to all such propositions themselves' [*PoM*, p. 449].[5] One might find it confusing to use the word 'term' to speak of an entity. But Russell does so presumably because he sticks with the traditional division of the subject matter of logic into *term* (or *concept*), *judgment*, and *inference* (cf. [*Papers 7*, p. 105]). Russell argues that, given the above definition of term, it is self-contradictory to suppose that something is not a term: the very statement 'something is not a term' turns that something into an object of thought, thus making it a term. Therefore 'terms embrace everything that occurs in a proposition' [*PoM*, p. 46].

The terms are then divided into *things* on the one hand, and *concepts* on the other hand [*PoM*, pp. 54–55]. Russell takes things to be terms that always 'occur as subject' in a proposition, whereas concepts can also occur in other ways. For example, '[i]n "Socrates is human", the notion expressed by *human* occurs in a different way from that in which it occurs when it is called *humanity*, the difference being that in the latter, not in the former, the proposition is *about* this notion' [*PoM*, p. 45].

Furthermore, Russell divides concepts into those expressed by adjectives and those by verbs [*PoM*, p. 44]. The former are unary properties such as *humanity*, while Russell argues that the latter, including those given by intransitive verbs, may all be regarded as relations [*PoM*, p. 49]. An important difference between these two kinds of concepts lies in the fact that the former are said to have instances, whereas this cannot be said about the latter [*PoM*, pp. 51–52].

Russell draws yet another distinction among terms: he divides the concepts indicated by adjectives into *predicates* and *class-concepts*. For example, the word 'human' is said to express a predicate, where Russell somewhat sloppily uses the word 'predicate' to speak of an entity. Russell also remarks, 'Predicates are, in a certain sense, the simplest kind of concepts, since they occur in the simplest type of proposition [*PoM*, p. 55]. On the other hand, the word 'man' is an expression for a class-concept. One may well wonder if the distinction is anything more than a verbal one. Russell indeed concedes that there may be little in this distinction. He remarks, of the predicate *human* and the class-concept *man*, that the latter 'differs little, if at all, from the predicate' [*PoM*, pp. 55–56].

[5]Russell thus also views propositions as terms. Each proposition is in his view composed of other entities and yet possesses a unity that makes it one.

These distinctions among terms might look somewhat trivial. However, Russell introduces them because he thinks they play fundamental roles in his account of key notions in logic and mathematics [*PoM*, p. 45]:

> One predicate always gives rise to a host of cognate notions: thus in addition to *human* and *humanity*, which only differ grammatically, we have *man*, *a man*, *some man*, *any man*, *every man*, and *all men*, all of which appear to be genuinely distinct one from another. The study of these various notions is absolutely vital to any philosophy of mathematics; and it is on account of them that the theory of predicates is important.

In the next section we look into how Russell explains, in terms of one of these notions, what *variables* are.

4 Russell's early account of variables

In *PoM* Russell articulates what may be called the *theory of denoting concepts*. Russell maintains that expressions of the form 'all A', 'every A', 'any A', 'an A', and 'some A', where 'A' is an expression for a class-concept, stand for what he calls *denoting concepts*. These concepts are said to have the relation of *denoting* to certain combinations of terms. For example, denoting concepts of the form *all A* are said to denote a *numerical conjunction* of entities. Such a conjunction appears, for instance, in the proposition "Brown and Jones are two of Miss Smith's suitors."[6] This proposition is about Brown and Jones collectively, and not about Brown or about Jones respectively. Russell takes this as a characteristic of a numerical conjunction. As for denoting concepts of the form *any A*, Russell maintains that they denote a *variable conjunction* of terms. Russell illustrates this notion using the proposition "if it was Brown or Jones you met, it was a very ardent lover." This proposition is equivalent to the conjunction of "if it was Brown you met, it was a very ardent lover" and "if it was Jones you met, it was a very ardent lover."

Russell also gives the following characterisation of the differences in denotation of concepts expressed (in the context of a proposition) by expressions of the form 'all A', 'every A', and 'any A' [*PoM*, pp. 58–59]:

[6]Since on Russell's view a *proposition* is an entity, but not a sentence, we use (as Russell himself does) double quotation marks to speak of the former, using single quotation marks for quoting sentences.

> *All a's*, to begin with, denotes a numerical conjunction; it is definite as soon as *a* is given. The concept *all a's* is a perfectly definite single concept, which denotes the terms [i.e., entities] of *a* taken all together. The terms so taken have a number, which may thus be regarded, if we choose, as a property of the class-concept [*a*], since it is determinate for any given class-concept. *Every a*, on the contrary, though it still denotes all the *a*'s, denotes them in a different way, *i.e.* severally instead of collectively. *Any a* denotes only one *a*, but it is wholly irrelevant which it denotes, and what is said will be equally true whichever it may be. Moreover, *any a* denotes a variable *a*, that is, whatever particular *a* we may fasten upon, it is certain that *any a* does not denote that one; and yet of that one any proposition is true which is true of any *a*.

It seems that Russell sees no difference between a concept *any a* denoting a variable conjunction of terms and its denoting 'only one *a*' where 'it is wholly irrelevant which it denotes'. Yet it is unclear how these two notions can be identified, in particular because a variable conjunction and the other four combinations of terms are said to be 'neither terms nor concepts' but 'strictly and only combinations of terms' [*PoM*, p. 58].[7]

At any rate, this view of the denotation of *any a* is intimately connected with the account of variables that Russell formulates in [*PoM*, chapter VIII].[8] It is worth quoting Russell somewhat at length here. He starts out as follows [*PoM*, p. 90–91]:

> Originally, no doubt, the variable was conceived dynamically, as something which changed with the lapse of time, or, as is said, as something which successively assumed all the values of a class. This view cannot be too soon dismissed. If a theorem is proved concerning *n*, it must not be supposed that *n* is a kind of arithmetical Proteus, which is 1 on Sundays and 2 on Mondays, and so on. Nor must it be supposed that *n* simultaneously assumes all its values. If *n* stands for any integer, we cannot say that *n* is 1, nor yet that *n* is 2, nor yet that it is

[7] Russell employs the word 'object' 'to cover both singular and plural, and also cases of ambiguity, such as "a man"' [*PoM*, p. 55]. We shall italicize the word when we use it in this way.

[8] In *PoM* and elsewhere, Russell employs the word 'variable' to speak both of such symbols as '*x*' and of what he thinks they stand for. In what follows, we shall use the word exclusively for the former, employing the phrase 'variable object' for the latter.

> any other particular number. In fact, n just denotes *any* number, and this is something quite distinct from each and all of the numbers. It is not true that 1 is any number, though it is true that whatever holds of any number holds of 1. The variable, in short, requires the indefinable notion of *any* which explained in Chapter v.

Russell thus thinks that the notion of variable must be explained in terms of the notion of *any*. He then moves on to explain the 'true variable' and the other variables—or what he calls 'restricted variables'—as follows [*PoM*, p. 91]:

> We may distinguish what may be called the true or formal variable from the restricted variable. *Any term* [i.e., *any entity*] is a concept denoting the true variable; if u be a class not containing all terms, *any* u denotes a restricted variable. The terms included in the object denoted by the defining concept of a variable are called the *values* of the variable: thus every value of a variable is a constant. There is a certain difficulty about such propositions as "any number is a number." [...] [I]f "any number" be taken to be a definite object, it is plain that it is not identical with 1 or 2 or 3 or any number that may be mentioned. Yet these are all the numbers there are, so that "any number" cannot be a number at all. The fact is that "any number" denotes one number, but not a particular one. [...]

In this way Russell proposes to understand the variable (object) as the *object* the concept *any term* denotes. He speaks of *the* variable (object) because it is a unique object—the variable conjunction of all the terms.

Russell thus holds that a variable expresses the denoting concept *any term*, which in turn denotes what he calls 'the variable'—the variable conjunction of *terms*. However, Russell makes a further move by recognising that *correlation* of values of variables presents a complication [*PoM*, p. 94]:

> Thus x is, in some sense, the object denoted by *any term*; yet this can hardly be strictly maintained, for distinct variables may occur in a given proposition, yet the object denoted by *any term*, one would suppose, is unique. This, however, elicits a new point in the theory of denoting, namely that *any term* does not denote, properly speaking, an assemblage of terms, but denotes only one term, only not one particular definite term. Thus *any term* may denote different terms in different places. We

> may say: any term has some relation to any term; and this is quite a different proposition from: any term has some relation to itself. Thus variables have some kind of individuality. [...] A variable is not *any term* simply, but any term as entering into a propositional function.[9]

So Russell now speaks of '(unrestricted) variables' rather than 'the variable' and maintains that they have 'some kind of individuality' arising from their *contexts of use*, or more precisely, from the propositional functions in which those concepts occur.[10] Moreover, in the last sentence of this passage he seems to mean by 'variables' denoting concepts rather than what they denote. This is probably because Russell now thinks there is no such thing as the unique object that the concept *any term* always denotes: '*any term* may denote different terms in different places.'

In this way, though Russell first holds that such symbols as 'x' express the denoting concept *any term*, which denotes what he calls 'the variable'—the variable conjunction of terms, he goes on to endorse a contextualist account of variables: *any term* denotes different entities in different contexts, that is, in different propositional functions. But Russell ends his discussion in *PoM* by conceding that all this does not amount to a fully worked out theory of variables as they are used in mathematics [*PoM*, p. 94].

5 Comparison between Russell's theory and Fine's theory

Russell's view of variables in *PoM* is not developed in as much detail or articulated as clearly as Fine's theory of arbitrary objects. Indeed, from the epigraph to this article we see that Russell freely admits that he has not arrived at a settled and satisfactory view. But it is clear that Russell's tentative account in *PoM* has deep similarities with Fine's account of arbitrary objects. In this section we compare these views.

[9]In this passage Fine finds the 'antinomy of the variable': to put it crudely, two variables occurring in a single sentence seem to have different meanings, whereas two variables occurring in different sentences seem to have the same meaning [Fine 2003]. Fine proposes an alternative semantics for a first-order language, putting forward *relational relationism*. Pickel and Rabern, who offer another way to resolve the antinomy, also hold that Russell was indeed confronted with this antinomy [Pickel & Rabern 2016].

[10]Russell explains—though tentatively—a propositional function as what we would obtain by replacing an entity in a proposition by the denoting concept *any term* [*PoM*, p. 84]. See also [*PoM*, pp. 92–93]. It is widely accepted in the literature that propositional functions (or what Russell calls thus) are not universals. See, for example, [Landini 1998].

As we have seen in section 1, Fine holds that Russell is critical of the notion of arbitrary object [Fine 1983, p. 55], [Fine 1985, p. 5]. But the contrary is the case. Fine refers to the following passage [*PoM*, p. 91]:

> By making our x always an unrestricted variable, we can speak of *the* variable, which is conceptually identical in Logic, Arithmetic, Geometry, and all the other formal subjects. The *terms* dealt with are always *all* terms; the complex concepts that occur distinguish the various branches of Mathematics.

Yet, this is where Russell elaborates on his view that the concept *any term* denotes what he calls 'the variable' or 'true variable'—the variable conjunction of all *terms*. And aspects of the theory of denoting concepts, which underlies this view, are comparable to Fine's theory of arbitrary objects. On the former view, a denoting concept *any F* denotes all *F*'s, while, on the latter, an arbitrary object *F* has all specific *F*'s as its value-range.

This structural similarity is not lost when Russell goes on to entertain a contextualist account of variables. In this view he no longer thinks that there is a unique object called 'true variable' and instead calls various occurrences of the denoting concept *any term* 'variables'. But he still maintains that a denoting concept *F* denotes an arbitrary *F*.

It should also be noted that Russell's notion of denoting is a relation between objects, not a semantic relation that *symbols* bear to their meanings. In *PoM* Russell maintains, as we have already seen in section 3, that denoting concepts are entities—or, in his terminology, *terms*. Russell also makes it explicit that the relation of denoting is not a linguistic one but one holding between a concept and a combination of terms [*PoM*, p. 47]:

> [...] meaning, in the sense in which words have meaning, is irrelevant to logic. But such concepts as *a man* have meaning in another sense: they are, so to speak, symbolic in their own logical nature, because they have the property which I call *denoting*.

Russell here sharply distinguishes between the relation between words and their 'meanings' on the one hand, and the relation between denoting concepts and what they denote on the other hand, dismissing the former relation as 'irrelevant to logic'.

There is also a similarity that Russell's initial account of variables shares with Fine's theory of arbitrary objects, but not with Russell's own modified, contextualist account. Russell initially held that 'any *F*' stands for *the* arbitrary object *F*, that is, the denoting concept *F*. So for any kind (or class-concept) F, there is, also on Russell's view, what Fine calls the unique independent arbitrary F. However, Russell moves on to acknowledge that for any given kind F, there apparently can be more than one variable object the range of which is all of F: we have seen Russell distinguish between the arbitrary F on the one hand, and 'the arbitrary F as entering into a propositional function' on the other hand. Objects of the latter kind are what, in his *PoM* view, *variables* ultimately are.

One may indicate a point of difference between Russell's theory of denoting concepts and Fine's view of arbitrary objects. On the former, a denoting concept itself may *not* be among the terms it denotes, whereas on the latter an arbitrary object itself possesses the characteristic property in question. For example, Russell, as we have seen, holds that '"any number" cannot be a number at all', while according to Fine an arbitrary number is a number.

Yet at the same time it appears that Russell is groping towards Fine's distinction between generic and classical conditions when he says that '"any number" cannot be a number at all [..., and] that "any number" denotes one number, but not a particular one.' In the first part of this statement, the classical reading of the condition 'being a number' is intended; in the second part of the sentence, a generic reading of 'being a number' is taken.

We believe that the most important difference between Russell's *PoM* view and Fine's view lies in their respective treatments of dependence relations between variables. Unlike Fine, Russell does not stress the key importance of the notion of *dependence* in arbitrary object theory.

6 Russell's subsequent views on variable objects

Even though Russell does not develop the notion of dependence between arbitrary objects in *PoM*, he does so after the completion of *PoM*. Russell then makes various attempts to find a workable formal system, and these efforts culminate in *Principia Mathematica* [*PM*]. In this section we restrict ourselves to giving a very rough sketch of the way in which Russell develops the notion of dependence between variable objects in the intermediate period from 1903 to 1910.

After the completion of *PoM* in December 1902 Russell goes on to endorse, albeit tentatively, the idea that *functions* or dependent variables are objects that are dependent on individual variable objects. In his 1903 manuscripts, we find two distinct accounts of what he calls 'propositional functions'. One is the view that a propositional function is an entity that we can obtain by *detaching* it from a proposition, which itself is considered a complex entity. Russell discusses this view mainly in a 1903 manuscript entitled 'Dependent Variable and Denotation'. Yet he soon abandons it because he finds that once propositional functions are treated as detachable entities—as *terms* or what he now calls *individuals*—he cannot avoid a functional variation of Russell's paradox. He then considers another view of propositional functions—the view that they are complexes *containing* a variable object. In his manuscript 'On Meaning and Denotation', Russell invokes this view to explain why functions are now to be put into a hierarchy: no dependent variable object can be, he argues, among the values of the individual variable object—ranging over individuals—because 'all the values of the individual variable are constants' [*Papers 4*, p. 333]. Hence 'if we want a variable whose values are to be dependent variables containing x, we must have a new variable of a different kind' [*Papers 4*, p. 334]. Russell thus comes to view propositional functions as dependent on individual objects.

We may compare this view of functions as dependent on other objects to Fine's notion of dependent arbitrary object. Both have in common the idea that what dependent variables stand for are objects that depend on other objects. A point of difference is that Russell is committed to the idea that a propositional function is a complex *containing* an independent variable (conceived as an object), whereas Fine does not take dependent arbitrary objects to contain independent arbitrary objects.

This point of difference is directly related to a problem that is peculiar to the view of a propositional function as a complex containing a variable object. The problem is that the view in question presupposes (independent) variables, while Russell finds it necessary to account for variables in terms of propositional functions. As we have seen, in *PoM* Russell comes to think of a variable object as '*any term* occurring in a proposition'. In 'On Meaning and Denotation', Russell observes that the individuality of a variable presupposes that 'an expression containing x must be treated as a whole, and must not be regarded as analysable into bits each of which contains an independent variable' [*Papers 4*, p. 333]. Yet, if variables are to be 'treated as a whole' and explained in terms of propositional functions, and if propositional functions are explained as a complex containing a variable object, we would be left with a circular account. This problem does not arise in Fine's account because he rejects the second antecedent. Fine does not claim that any dependent arbitrary object contains at least one independent arbitrary object as a part.

Russell's 'On Denoting' is famous for its contextual analysis of definite descriptions, which translates a sentence 'C(the F)' into one without any occurrences of the definite description: '$\exists x(\mathrm{C}x \,\&\, \forall y\,(\mathrm{F}y \leftrightarrow x = y))$'. But the theory presented in this article is also concerned with other kinds of denoting expressions. It effectively replaces his old theory of denoting concepts with a straightforward translation of natural language sentences into those of a first-order language. For example, '"C(all men)" means "'If x is human, then C(x) is true' is always true"' [Russell 1905, p. 481].[11]

It may thus appear that Russell abandons the whole apparatus of denoting when he hits upon the celebrated theory of denoting expressions in 'On Denoting'. This is not true, however. The theory presented in 'On Denoting' simply reduces the whole problem of denoting into that of a variable because it uses individual variables. In his letter to Russell of 23 October 1905, Moore asks Russell: 'Have we, then, immediate acquaintance with the variable? and what sort of entity is it?' [*Papers 4*, xxxv]. Russell admits, in his reply, that 'the question [...] about the variable is puzzling', and goes on to say: 'I only profess to reduce the problem of denoting to the problem of the variable' [*Papers 4*, xxxv]. In fact, in a manuscript 'On Fundamentals', written prior to 'On Denoting', Russell remarks: 'The interesting point [...] is that *any* is genuinely more fundamental than other denoting concepts; they can be explained by it, but not it by them' [*Papers 4*, p. 387]. He adds, '*any* itself is not fundamental in general, but only in the shape of *anything*', where '*Anything* seems to be exactly the same as *the variable*' [*Papers 4*, p. 387].

By the time when he writes *PM* with Whitehead, Russell has come to view, as Frege did, a variable as 'any *symbol* whose meaning is not determinate' [*PM*, p. 4; emphasis added]. By thus adopting the *linguistic account of logical variables*, Russell can indeed avoid the problem of the individuation of the (independent) variable: he no longer needs to appeal to the notion of arbitrary objects to account for variables.

With this quiet move, we may appear to have reached an anticlimactic conclusion to Russell's quest for the metaphysical nature of the variable. The linguistic account has become the received view, which is captured well by Quine, fifty years later [Quine 1975, p. 155]:

> It used to be necessary to warn against the notion of variable numbers, variable quantities, variable objects, and to explain that the variable is purely a notation, admitting only fixed numbers or other fixed objects as its values. This dissociation now seems to be generally understood, so I turn to others.

Yet it may still be possible to find in *PM* another instance where Russell

[11] The use of the truth predicate is of course not essential here.

envisages an idea similar to Fine's notion of dependence among arbitrary objects.[12] In *PM*, Whitehead and Russell endorse the view of a propositional function as an ambiguity. On this view, a propositional function 'ambiguously denotes' some one of the given objects—its values. This is precisely what they think of as 'The Nature of Propositional Functions' [*PM*, p. 38]. The authors then go on to remark [*PM*, p. 39]:

> A function is what ambiguously denotes some one of a certain totality, namely the values of the function; hence, this totality cannot contain any members which involve the function, since, if it did, it would contain members involving the totality, which, by the vicious-circle principle, no totality can do.

Whitehead and Russell thus argue that propositional functions form a hierarchy, thereby subject to the restrictions of orders and types. *If* we may understand propositional functions in *PM* as *objects*, then their *PM* theory still contains traces of the theory of variable (or arbitrary) objects.

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[12] We say 'may' because the following interpretation is contentious, to say the least. Prominent scholars, including Gregory Landini, Graham Stevens, and Kevin C. Klement, maintain that what Russell and Whitehead call propositional functions are not objects of any kind but open well-formed formulae [Landini 1998, Stevens 2005, Klement 2010]. Although their interpretations merit detailed discussion, due to limitation of space we must be content in this essay merely to indicate a possible alternative.

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