PER ASPERA AD ASTRA: FROM SKOLEM PARADOX TO AN UNCOUNTABLE UNIVERSE

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Abstract. In this article we argue in favour of the existence of uncountable collections. Specifically, we will argue that the universe of set theory is uncountable. The argument is based on the analysis of Skolem Paradox and moves from its premises and from a comparison between Cantor Theorem and Cohen Theorem about the existence of generic filters. We then address an iterated version of the skeptic argument, outlining an important role that Hartogs Theorem can play in this respect. This paper also aims to connects the criticisms of the uncountable based on Skolem Paradox and the more recent discussion on Countabilism: the position according to which everything is countable.

Introduction

During the 80s Löwenheim-Skolem Theorems and Skolem's Paradox were at the centre of a debate about the indeterminacy of reference of formal languages, originated from Putnam's model theoretic argument [22]. This debate echoed some important criticism that Skolem himself voiced against set theory, some sixty years before Putnam. Without entering an exegetical discussion, Skolem viewed the antinomy that still bears his name as suggesting a relativity of the fundamental notions of set theory, among which that of being uncountable. Of course there are important differences between Putnam's and Skolem's arguments, but they both belong to the same skeptical tradition. In the wake of Skolem's skepticism a few authors have recently considered the possibility (or better the consistency of the statement) that there are no uncountable collections [8, 20, 21, 23]. This new approach, named Countabilism, however, does not explicitly take their move from Skolem Paradox, but from an analogy of Cantor's Theorem and Russell's paradox.

The aim of this paper is twofold. On the one hand we want to connect the old and the new criticisms to the uncountable, displaying the different strategies of attack they put forward. We will see that they compose a coherent, although heterogeneous, family kept together by a common distrust of anything that goes beyond countable. On the other hand we want to defend the existence of uncountable collections and therefore respond to these criticisms. In order to do so, we will show how the same formal results that give rise to Skolem's Paradox (and some more) can be used against this skeptical tradition. We will argue that we can defend that the universe of all sets is uncountable. Our arguments are rooted on a theory/meta-theory distinction that, of course, cannot be overcome in a unique formal system. Therefore, we cannot formally prove that Skolemite is wrong and we have to accept that it is consistent to assume that everything is countable. However, by reflecting on the interplay between syntax and semantics we can realise that our domain of the set-theoretical discourse is uncountable and therefore that it is worth studying with formal systems like ZFC. We will defend that, although consistent, assuming that everything is countable has the defect of non faithfully depicting the universe of set theory. If something can be learned from Löwenheim-Skolem Theorems, it is exactly that we cannot believe everything a model tells us about reality.

The paper is structured as follows. In Section §1 we review Skolem's Paradox and Cantor's Theorem. Then, Section §2 is devoted to show that Cohen's Theorem about the existence of generic filters bears important similarities with Cantor's Theorem at a meta-theoretical level. In §3, we sum up and lay down in details the skeptical arguments against the notion of uncountable. Finally, section §4 will provide our responses to the skeptical arguments. We end in §5 with a few concluding remarks.

1. Skolem's Paradox

Skolem's Paradox is not a paradox, but only a counter-intuitive consequence of Löwenheim-Skolem Theorem. In order to explain why, we first need to introduce Cantor's Theorem.

Theorem 1. ***Cantor's Theorem (CaT)*** *Given a set X, the cardinality of its power-set $\mathcal{P}(X)$ exceed that of X; i.e. $|X| < |\mathcal{P}(X)|$. In other terms, there is no bijection f able to univocally associate the elements of X with all elements of $\mathcal{P}(X)$.* □

CaT represents the cornerstone of the theory of infinite cardinals, since its shows that infinity comes in different sizes. When applied to the set of natural numbers, $\mathbb{N}$, CaT shows that its power-set has a greater infinite cardinality. Moreover, since $\mathcal{P}(\mathbb{N})$ can easily be put into a one-to-one correspondence with $\mathbb{R}$, CaT shows that two infinite sets at the centre of mathematical investigation: $\mathbb{N}$ and $\mathbb{R}$, have different cardinalities.

It is instructive to analyse the standard diagonalisation argument used to prove CaT. For the sake of concreteness, let us consider the case of natural numbers. In order to show that $|\mathbb{R}| > |\mathbb{N}|$, we assume that we have an enumeration (without repetitions)[1] of all real numbers in length ω, say $(a_n)_{n\in\omega}$. Since $\mathbb{R}$ has the same cardinality of $\mathbb{R} \cap (0,1)$ we can also assume that the a_n's are of the form $0, a_n^1 a_n^2 a_n^3 \ldots$ We can picture this enumeration as in Table 1, where we just ignore the integer part.

Now, we claim that we can generate a new real number that is not displayed in the enumeration $(a_n)_{n\in\omega}$. For every a_n consider its n-th digit. If $a_n^n \in \{x \in \mathbb{N} : 0 \leq x \leq 8\}$ then let b_n be the number $a_n^n + 1$. Otherwise, if $a_n^n = 9$, then $b_n = 8$. It is easy to see that the number $b = 0, b_1 b_2 b_3 \ldots$ is different from any a_n, since its n-th digit is different from a_n^n by construction. However, b is a real number. Therefore, we conclude that the enumeration $(a_n)_{n\in\omega}$ did not include all real numbers. Since we can produce the same

[1]In particular, we identify an infinite sequence of consecutive 9s with the upper bound of all its initial segments.

	1	2	3	4	5	...	n	...
a_1	a_1^1	a_1^2	a_1^3	a_1^4	a_1^5	...	a_1^n	...
a_2	a_2^1	a_2^2	a_2^3	a_2^4	a_2^5	...	a_2^n	...
a_3	a_3^1	a_3^2	a_3^3	a_3^4	a_3^5	...	a_3^n	...
a_4	a_4^1	a_4^2	a_4^3	a_4^4	a_4^5	...	a_4^n	...
a_5	a_5^1	a_5^2	a_5^3	a_5^4	a_5^5	...	a_5^n	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
a_n	a_n^1	a_n^2	a_n^3	a_n^4	a_n^5	...	a_n^n	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$

TABLE 1. Countable enumeration of reals.

argument for any possible countable enumeration, this shows that there is no such enumeration.

It is important to notice that this diagonal argument does not show directly that the reals are uncountable, but only that any countable enumeration is incomplete. In order to complete the argument we need to know that the reals do form a set, with a fixed cardinality. If that is the case, then the set of reals has an uncountable cardinality. On the other hand, if the reals do not form a set, CaT still holds, but no consequences can be drawn about their cardinality.[2]

Let us now move to Löwenheim-Skolem Theorem. There are at least two versions of it: a downward version and an upward version. The former provides the existence of "smaller" models, while the latter of "larger" models. We state a particular version that perfectly fits the present discussion. However, we stress that LST represents a general feature of any first order theory.

Theorem 2. ***Löwenheim-Skolem's Theorem (LST)*** *If* ZFC *has a transitive well-founded model, say* $\mathcal{M}$*, then we can find a countable transitive elementary substructure* $\mathcal{N}$ *of* $\mathcal{M}$ *(*$\mathcal{N} \preccurlyeq \mathcal{M}$*).*

The hypotheses that the starting model is transitive and well-founded are not needed for the countability of the final model (and indeed these conditions are not needed in the most general form of LST). Their role is only ensure that the models we are dealing with are not pathological and un-intended, but they closely resemble the universe of all sets **V**: a transitive class that validates the Foundation Axiom.

Now, Skolem Paradox ensues from noticing that a transitive countable model of ZFC still validates CaT and therefore models the existence of uncountable sets. Consider the specific case of $\mathbb{R}$. A model like the one provided by Theorem 2 contains a set $\mathbb{R}^{\mathcal{N}}$ that $\mathcal{N}$ considers to be the set of all reals. By transitivity, all elements of $\mathbb{R}^{\mathcal{N}}$ belong to $\mathcal{N}$, that is $\mathbb{R}^{\mathcal{N}} \subseteq \mathcal{N}$.[3]

[2]For a criticism of CaT along these lines see [12] and [20].

[3]This is why the transitivity of the model is important for Skolem Paradox (although not essential for LST). Indeed, in its general form, LST can produce a countable model to which $\mathbb{R}$ (the *real* $\mathbb{R}$) belongs as an element, but of course not as a subset. In this

Moreover, by countability of $\mathcal{N}$ the set $\mathbb{R}^{\mathcal{N}}$ is countable. But how is this possible? Indeed, we have

$$\mathcal{N} \models \mathbb{R}^{\mathcal{N}} \text{ is uncountable.} \tag{1}$$

The paradox is easily dissolved if we apply to the present case the definition of what it means to be uncountable. Indeed, "$\mathbb{R}^{\mathcal{N}}$ is uncountable" is translated as "$\neg\exists f : \mathbb{R}^{\mathcal{N}} \to \mathbb{N}$, with f bijective". Therefore, (1) just shows that $\mathcal{N}$ does not contain a bijection between $\mathbb{R}^{\mathcal{N}}$ and ω, witnessing the countability of $\mathbb{R}^{\mathcal{N}}$. However, when we argued before that $\mathbb{R}^{\mathcal{N}}$ was countable, we did not argued within $\mathcal{N}$ –or, as it is normally said, from the perspective of $\mathcal{N}$– but in the meta-theory of $\mathcal{N}$. The latter is whatever theory that is able to prove Theorem 2. In such meta-theory –that happens to be ZFC by its incredible expressive power– we can prove that there is a bijection between $\mathbb{R}^{\mathcal{N}}$ and ω. The fact that $\mathcal{N}$ models "$\mathbb{R}^{\mathcal{N}}$ is uncountable" only tells us that this bijection does not belong to $\mathcal{N}$. This is the solution of Skolem Paradox: the bijection witnessing the countability of the object, that a model thinks is uncountable, does not belong to that model. This is why Skolem Paradox is not a paradox. The paradoxically only originates from a confusion of levels between theory and meta-theory; between the theory of a model and the meta-theory in which we prove the (possibly conditional)[4] existence of such a model.

Skolem Paradox has been used to argue that the notion of uncountable collection is relative, in the sense that there is no absolute such notion and any attempt to pin it down in an absolute way is doomed to fail. Someone who maintains such a position can be called Skolemite, although it is controversial whether Skolem actually was a Skolemite.[5] There is a vast literature on Skolemites and their near cousins, the Putnamites; after Putnam's [22]. In a nutshell, Putnam's argument was based on Löwenheim-Skolem Theorems and aimed to show that formal theories, alone, cannot fix their reference; the relativity of the notion of uncountable, suggested by Skolem Paradox is only a component of the relativity of reference argued by Putnam. Skolemites' and Putnamites' arguements have been challenged on formal grounds [3, 13] and from from a realist [7, 16], a pragmatic [11], and descriptivist perspective [24]. It has also been suggested that the Skolemite's position is self-refuting [9]. Our arguments will draw on the literature, but now with a positive twist. Indeed, we will argue that not only the Skolemite's position is untenable, but also that once we accept the formal results on which the

case one could easily dismiss the paradox by saying that such a model contains the reals only nominally and not substantially. Such a model would be pathological by such low standard that it would not threat any reasonable understanding of set-theoretic notions. See [7] for a discussion on this point.

[4]Notice that Theorem 2 is in a conditional form. This is how we can talk about the existence of a model of ZFC *in* ZFC. We can of course prove the existence of such a model in a strictly stronger theory. However, using ZFC also in the meta-theory helps to provide the sense of confusion on which Skolem Paradox thrives.

[5]See [12] and [7] for a discussion on this point.

Skolemite's criticism are based, the only reasonable conclusion is the acceptance of the uncountable. In order to do so we need another important piece of mathematics: Cohen's Theorem on the existence of generic filters.

2. Cohen's Theorem

Forcing is a technique introduced by Cohen, in 1963, to prove the independence of the Continuum Hypothesis. This method consists in expanding a model of ZFC, the *ground model*, to a larger one, the *generic extension*, which contains more sets and still validates ZFC. This construction can produce a model where $|\mathbb{R}| = \aleph_2$, thus showing the (relative) consistency of $\mathsf{ZFC} + \neg\mathsf{CH}$. Without entering the details of the construction, the idea is that the new sets of a generic extension are built by approximation using, as building blocks, the sets in the ground model. The collection of all these approximation is a partially ordered set $\mathbb{P}$ (a poset), that under some mild conditions –that of the posed being *separative*[6]– can always be use to produce a generic extension that property extends its ground model.

The elements of a poset represent all possible building block for the construction of a new set. For example, if we want to construct a new $X \subseteq \omega$, a real, we can consider the poset of all finite approximations of subsets of ω. Out of all these approximations we can select a blue-print for the construction of X. This blue-print is called a *generic filter* and it is normally indicated as G; or G_X to indicate that it is the blue-print of X. Then, a forcing construction consists in choosing a ground model $\mathcal{M}$ of ZFC to which a poset $\mathbb{P}$ belongs and to produce a generic extension $\mathcal{M}[G]$, still validating ZFC to which a generic filter $G \subseteq \mathbb{P}$ belongs. In this case, the generic extension $\mathcal{M}[G]$ is completely determined by G, in the sense that it is the smallest model of ZFC that contains both $\mathcal{M}$ and G.

There are two key aspects of this construction:

(1) if the model $\mathcal{M}$ is *countable*, with $\mathbb{P} \in M$, it is possible to prove the existence of an $\mathcal{M}$-generic filter[7] $G \subseteq \mathbb{P}$;
(2) if a poset $\mathbb{P}$, belonging to a countable model $\mathcal{M}$, is *separable*, then any $\mathcal{M}$-generic filter $G \subseteq \mathbb{P}$ is such that $G \in \mathcal{M}[G] \setminus \mathcal{M}$.

These are expressed by the fundamental theorem of forcing.

Theorem 3. ***Cohen's Theorem (CoT)*** *Given a separative partial order $\mathbb{P}$, belonging to a countable transitive model $\mathcal{M}$ of* ZFC*, and G a $\mathcal{M}$-generic filter, there is an model $\mathcal{M}[G]$ which is the smallest model of* ZFC *such that $\mathcal{M} \subseteq \mathcal{M}[G]$ and $G \in \mathcal{M}[G] \setminus \mathcal{M}$.*

CoT conveys many information, but what we would like to stress here is its analogy with CaT. They both can be seen as showing the incompleteness of countable objects. While CaT shows that any countable enumeration

[6]A partial order is called separative if any of its element has two extensions, in the sense of the order, that are not compatible; meaning that they do not have a common extension.
[7]For the present paper it is not essential to define the notion of $\mathcal{M}$-generic filter. Informally, it is a property of the filter that ensures its genericity from an intuitive perspective. If we consider the example of building a new real X, the fact that G_X is an $\mathcal{M}$-generic filter is what makes X an arbitrary real number from the perspective of $\mathcal{M}$. For a formal defintion of this notion see [15].

of $\mathbb{R}$ is incomplete and can be diagonalised in order to produce a new real number that did not belong to the previous enumeration, in a similar way, but now at a meta-theoretical level, CoT shows that any countable model of ZFC is incomplete, since it can be extended to a larger one that contains new sets (i.e. new generic filters). Therefore, if we see a model of ZFC as a (potentially partial) picture of the universe of sets $\mathbf{V}$, CoT tells us that any countable enumeration of all sets is incomplete. We can always find new sets that did not belong to that enumeration (i.e. model), by applying CoT.

It is probably now clear how we will use CoT to argue that $\mathbf{V}$ is uncountable, but let us provide a step-by-step argument, in order to clear its internal dialectic and the problem it causes to the Skolemite.

3. The main skeptical arguments

So far we have introduced formal results and only hinted on how they can be used in favour or against the uncountable. It is now time to see exactly how they figure in the arguments of the skeptics. For simplicity we will call them the Skolemites; again with the *caveat* not to read too much of Skolem in their positions. We can organise the arguments in a sequence that from a mild form of relativism arrives to a revisional stance towards the axiomatisation of set theory. Weaker or stronger forms of Skolem's Paradox can be identified in this master argument and many different existing positions can be fit there. However, the goal here is not exegetical and although we can hear echoing some ideas from Skolem or Putnam or Hamkins, however, we do not wish to claim that these are their arguments; even if this would be the case.

3.1. **First block: innocuous relativity.** We will put the arguments in a sequence of four blocks, loosely ordered by strength. The argument of each block can be proposed in isolation and is connected by family resemblance to the others. We can start from the innocuous juxtaposition of CaT and LST.

(1) LST provides us with countable transitive models of ZFC,
(2) CaT is provable in transitive countable models of ZFC,
(3) Therefore, the uncountability of a set, that derives from CaT, is relative to the model of ZFC under consideration.

In this crude form, this first step of the Skolemite argument only shows that the notion of being uncountable is relative to a model of ZFC; thus leaving open the possibility of finding a more robust such notion in the meta-theory. The reason why this first argument is not problematic is that it does not target the notions involved directly, but only their model-theoretic versions. Indeed, the semantics of formal theories is presented using model theory, which provides us with a clear interpretation of every mathematical concept. Then, if we accept that notions like truth and existence can only be interpreted in the context of a model, consequently it should not come as a surprise that also "being uncountable" receives the same treatment. Call it relativism if you wish, but it is only one provided by interpretation and that does not affect the same notions that are interpreted.

3.2. **Second block: absolute relativity.** Since our background meta-theory is also ZFC, then the innocuous argument can be repeated in the meta-theory. This is when the Skolemite argument starts to bite and to attack the very notion of uncountability. The Skolemites can indeed continue their argument in the following way.

(4) The best way to understand our meta-theory is through ZFC,
(5) Our meta-theoretical notion of uncountability is the same one displayed in a model of ZFC.
(6) Therefore, also our meta-theoretical notion of uncountability is relative.

The conclusion reached in (6) is stronger than the one reached in (3), since the relativity of the notion of uncountability is now pushed in the meta-theory. If the relativity of the notions that appear in a model is in principle harmless, since we can still compare them to a possibly "real" one in the meta-theory, the conclusion we reach in (6) is more problematic since it puts into question the absoluteness of the meta-theoretical notion of uncountability, collapsing it to the intra-model-theoretic one displayed by a model of ZFC. How can we measure the uncountability of a collection, if there is no absolute yardstick to which we can compare it?

This is the level where the Skolemites often develop Skolem's Paradox: they use CaT and LST to argue for the relativity of the theoretical notion of "being uncountable" and, then, the fact that ZFC is our meta-theory to argue that also our meta-theoretical notion of "being uncountable" falls short of being absolute. It is also at this level that we find Putnam's model theoretic argument: if our meta-theory is ZFC (: what set-theorists call, slightly abusing notion, the universe of all sets **V**), then the assumption of its consistency can be used, together with LST, to build a countable model that is indistinguishable from it, by any possible (theoretical or operational) means. Therefore our notion of intended model of ZFC, that is **V**, is underdetermined, since there is no formal means to distinguish it from a wisely cfrafted countable model.[8]

3.3. **Third block: the threat of iteration.** From the absolute relativity of the notion of "being uncountable" to argue that there are no uncountable collection, it is just a small step. This is the content of the next part of the Skolemite's argument.

(7) We only have a clear understanding of countable collections,
(8) Given the relativity of the notion of uncountability, every uncountable collection can be seen as countable from a higher perspective.
(9) Therefore, there are no absolute uncountable collections.

While steps (1)-(6) of the Skolemite's argument were directed against the concept of being uncountable, our Skolemites now become greedier and their criticism targets the objects themselves and not only the concepts. Indeed,

[8] In the case of Putnam's model-theoretic argument there is an important complication (that will not occupy us here) which consists in noticing that **V** is not, properly speaking, a model, but a proper class. Therefore it is not possible to apply LST to **V** directly. This is an important technical issue that has been raised in the literature [3, 4, 6] and that weakens significantly the conclusion of Putnam's argument.

steps (7)-(9) aim to attack the very existence of uncountable collections and not only the sharpness of our mathematical concept. Notice that this new block of the argument implicitly contains the germ of an iteration. This can be seen as follows: if the meta-theory of a model of ZFC, say $\mathcal{M}_0$, is just another model of ZFC, say $\mathcal{M}_1$, nothing prevents that this larger model can be seen to be countable from the perspective of an even larger model $\mathcal{M}_2$. And of course this idea can be iterated in order to always swallow up the (relative) uncountability displayed within a model by successive models, with respect to which the previous models become countable. Therefore, the Skolemite argues, there are no absolute uncountable collections, since every one we encounter is only such relative to the model we consider it in, but this uncountability will be destroyed by a larger perspective.

3.4. **Fourth block: Countabilism.** We are now on a slippery slope and once the skeptic doubts starts it is hard to be stopped. Indeed, the Skolemites could continue their disruptive path and push the argument to negate the very acceptance of ZFC as a correct formalisation of set theory.

(10) Since being uncountable is only relative to a model of ZFC, there are no absolute uncountable collections.
(11) Then, set theory should really be about countable collections.
(12) Therefore, we should axiomatise set theory accordingly: a theory where everything is countable.

This last bit of the argument extends the previous block by considering what happens at the limit of the iteration. If moving from one model to another can destroy the uncountability of a set, at the limit it will have destroyed every uncountable set and we will be left only with countable collections. This perspective has surfaced in the literature and recently has motivated an account called Countabilism, according to which everything is countable.

> Perhaps we would be pushed in the end to say that all sets are countable (and the continuum is not even a set) when at last all cardinals are absolutely destroyed.[9]
>
> Observing this situation and given our claim that there are not any really uncountable infinities, we might imagine ourselves as, so to speak, navigating an endless collection of these countable models in something like the generic multiverse we have described. While the illusion of multiple infinite cardinalities is witnessed inside each of the universes, we are free to move between them.[10]

As a consequence, defenders of Countabilism have a revisionary attitude towards set theory and propose new axiomatic systems able to account for a universe of sets where everything is countable. This, of course, calls for a modification of the formal tools, since ZFC and its Powerset Axiom will directly contradicts this view, due to CaT. This is way Countabilism needs to expand the logical resources and extend set theory with a modal language, as in [23], or change the notion of cardinality from the outset, as in [21].

[9]Dana Scott in the introduction to [5, p. xv].
[10] [20, p. 203].

The reason for putting Countabilism at last, in the context of a reconstruction of the Skolemite's argument, is the direct refutation of CaT. We started from a mild form of relativity and, by reflecting on the tension of CaT with LST, we arrived at an open opposition to the very theorem on which set theory is build, that is CaT, and the paradise of transfinite cardinalities that Cantor created for us.

Once we have laid out the different positions and arguments that belong to the Skolemite's tradition, it is now time to argue why these are legitimate but non-definitive arguments against the robustness of the notion of uncountability.

4. Addressing the skeptical arguments

We are now in the position to defend the notion of uncountable set from the charges of the Skolemites. To this aim we will address each block of the skeptical argument, showing how a careful reflection on the mathematical results involved suggest the exact opposite conclusions. Our responses will not rest on significant philosophical assumptions that can be disputed on their own ground, but only on a wider mathematical perspective able to dispel the doubts raised by a local perspective that misses a bigger, more multifaceted, picture.

4.1. **Answer to the second block: Cohen's Theorem.** We shall start from the second block, since the first step of the Skolemite argument does not really represent a skeptical doubt, but only the recognition of the local character of the notion of truth and existence on which a model-theoretic semantics is based. Remember that the second block loosely represents Skolem Paradox: the argument aimed to show that our meta-theoretical concept of "being uncountable" is as relative as our intra-theoretical notion, being both the notion provided by ZFC. This part of the skeptical argument is therefore based on the following assumptions: a) CaT, providing us with uncountable sets within a model of ZFC; b) LST, responsible to show the countability of these sets in the meta-theory, and c) the recognition that also our meta-theory is expressed in ZFC terms. Then, the conclusion on the absolute relativity of the notion of uncountable set follows by the collapse of the intra-theoretical and the meta-theoretical perspectives. This last step is, we believe, the mistake.

In order to see why, we only need to accept the first step of the skeptical argument: CaT shows that, within a model of ZFC, we have uncountable sets. As we noticed before, one could even dispute this conclusion, arguing, for example, that the reals do not even form a set, but a proper class. However, this interpretation of CaT is not available to the skeptical at this level of the argument, since otherwise Skolem Paradox does not even triggers. If we do not even have uncountable sets within a model of ZFC there is nothing odd in knowing that this model is countable. In conclusion, the Skolemites cannot avoid to accept that CaT gives us uncountable collections, or more precisely that the proof of CaT forces us to recognise that any countable collection of real numbers is incomplete; and therefore that the reals are uncountable.

This is where CoT comes in handy. Indeed, we can show that the intra-theoretical notion of uncountability, that was destroyed by LST, can be restored in the meta-theory by CoT. The argument is simple and is based on the analogy between CaT and CoT that we outlined in §2. As CaT shows that every countable enumeration of the reals is incomplete, in the same way CoT shows that every countable collections of elements of our meta-theory, that we can can arrange in the shape of a model of ZFC, is always incomplete. Therefore, if we accept that CaT tells us that the reals are uncountably many, in the same way we need to accept that our meta-theory is populated by uncountably many objects. In other words CoT tells us that universe of all sets is uncountable.

This is a fatal blow to any argument that attempts to collapse the intra-theoretical notion of uncountability with the meta-theoretical one. Indeed, even if our meta-theory is a structure satisfying ZFC as it is the case for the universe of all sets $\mathbf{V}$, still we have an important difference between the notion of "being uncountable in $\mathcal{M}$", where $\mathcal{M}$ is a model of ZFC and "being uncountable in $\mathbf{V}$". While the former can be shown to equivalent to "being countable in $\mathbf{V}$" by LST, the same cannot be done for the latter. Indeed, CoT will show that every attempt to show the countability of the meta-theory will always fail. And of course we cannot directly apply LST to $\mathbf{V}$, since this would directly contradiction Tarki's result on the undefinability of truth.

Consequently, the Skeptical argument of the second block is easily rebutted. It was meant to reduce the theoretical and meta-theoretical notions of uncountable collection, since are both grounded on ZFC, but CoT saved the day, by showing that we have an analogous result of CaT for the meta-theory: CoT. Therefore *there is* a notion of uncountable collection that is not swept away by LST, since no countable model of ZFC (notice that CoT applies to every countable set) will ever exhaust the universe of all sets.

4.2. **Answer to the third block: Hartogs Theorem.** Of course our job is not done and the Skolemites have still a few tricks in their hats. Indeed, the skeptic has available an important response to the above argument. Our defence of the uncountable was structured around two important points: first to keep adequately separated theory and meta-theory, and second to rely on the stability and unicity of the meta-theory. Although these two principles are rock-solid, the skeptic could cast doubts on our ability to have correctly individuated *the* meta-theory; and, with that, *the* correct notion of uncountable set. More concretely, a Skolemite could argue that what we called $\mathbf{V}$ in our previous argument, in the end, is not *the* meta-theory, but another set model of ZFC that we have mistakenly taken for all there is, but that itself is only a part of the universe of all sets. Therefore, LST can be applied to this structure and the Skolemite's argument can be repeated. And nothing prevents the Skolemites to iterate their strategy with respect to any possible new "$\mathbf{V}$" we can come up with.

Our response to this iterated argument has the form of a self-refutation and it is based on the recognition that also the Skolemites need to base their argument somewhere, on the pain of unintelligibility of their own position. In this sense, our argument is similar to the one presented in [9], where it is argued that in order to differentiate between the intra-theoretical and meta-theoretical notion of uncountability –a difference used to show that uncountability is a relative notion– we need to grasp sufficiently well the the meta-theoretical notion of uncountability. But this would contradict the main goal of the Skolemite: the impossibility to pin down this notion. Hence, the self-contradicting charge against the Skolemite's argument. In a similar vain we stress here that the iterated argument of the Skolemite needs to be phrased in a fixed context, intelligible to the Skolemite. But in this context we will find that it is possible to argue for the existence of uncountable collections using Hartogs Theorem. We first present this theorem and then we show how to use it for our purpose.

Theorem 4. ***Hartogs Theorem (HT)*** *Let X be a set, then* $\mathrm{Hrtg}(X) = \{\alpha \in \mathbf{Ord} | \exists f : \alpha \rightarrowtail X\}$ *(i.e. the collection of ordinals that injects in X, called the* Hartogs number of X*) is the smallest ordinal that does not inject into X and it is a cardinal.*

Proof. Given a set X se can define A_X as follows

$$A_X = \{(\alpha, f) | \alpha \in \mathbf{Ord} \wedge f : \alpha \rightarrowtail X\}.$$

For each $(\alpha, f) \in A_X$ we can let $W_{(\alpha,f)}$ be the well-order on $ran(f) \subseteq X$ induce by f, that is

$$xW_{(\alpha,f)}y \iff f^{-1}(x) \leq f^{-1}(y).$$

Therefore, $f : \langle \alpha, \leq \rangle \to \langle ran(f), W_{(\alpha,f)} \rangle$ is an isomorphism. If $(\alpha, f), (\beta, g) \in A_X$ and $W_{(\alpha,f)} = W_{(\beta,g)}$, then $g^{-1} \circ f : \langle \alpha, \leq \rangle \to \langle \beta, \leq \rangle$ is an isomorphism, hence $\alpha = \beta$ and $f = g$. In other words, the function

$$\Phi : A_X \to \mathcal{P}(X \times X)$$

such that $\Phi(\alpha, f) = W_{(\alpha,f)}$, is injective. As a consequence, the set A_X is a set, thanks to the Replacement and Powerset axioms. Moreover, the projection on its first coordinate is also a set.

$$\{\alpha \in \mathbf{Ord} | \exists f : \alpha \rightarrowtail X\}$$

Hence $\mathrm{Hrtg}(X)$ is a set. It is clear that $\mathrm{Hrtg}(X)$ is a transitive set, since the elements of an ordinals are ordinals as well and moreover if $\alpha \in \mathrm{Hrtg}(X)$ and $\beta \in \alpha$, then also β injects in X, as witnessed by the restriction to β of the f that forces α to belong to $\mathrm{Hrtg}(X)$. Since $\mathrm{Hrtg}(X)$ is a transitive set of ordinals, it is an ordinal.

In order to see that $\mathrm{Hrtg}(X)$ is the smallest ordinal that does not embeds into X, notice that if that was not the case, then $\mathrm{Hrtg}(X) \in \mathrm{Hrtg}(X)$, contradicting the Foundation axiom. Finally, to show that $\mathrm{Hrtg}(X)$ is a cardinal, suppose, towards a contradiction, that it was not. Then $|\mathrm{Hrtg}(X)| \in \mathrm{Hrtg}(X)$. But since $|\mathrm{Hrtg}(X)|$ is the cardinal of $\mathrm{Hrtg}(X)$, then there is a bijection between these two sets. In particular $\mathrm{Hrtg}(X) \rightarrowtail |\mathrm{Hrtg}(X)| \rightarrowtail X$, where the second injection is provided by the fact that $|\mathrm{Hrtg}(X)| \in \mathrm{Hrtg}(X)$. Then, composing the injective functions we would have that there is an injection of $\mathrm{Hrtg}(X)$ in X: a contradiction.

□

When X is an infinite ordinal $\alpha \geq \omega$, then

$$\text{Hrtg}(X) = \bigcup \{\beta | |\beta| = |\alpha|\} = \{\beta | |\beta| \leq |\alpha|\}$$

is the smallest cardinal strictly larger that α and it is denoted by α^+. The cardinal ω^+ is denoted by ω_1 and it is the least uncountable cardinal.

We can now go back to the iteration objection. The Skolemites pushed the relativity of the notion of uncoutability from the meta-theory of the model to which we applied LST to a larger meta-theory in which the first meta-theory lived; and continued to do so for every new meta-theory we can come up with. But how far can they go? This is the a question that the Skolemites should be able to answer in their own terms. This brings up to a second question, which is essential to the first: where does the iteration argument takes place?

Well, the Skolemite's argument takes place in the Skolemite's yard, whatever this is. In absence of a better word, let us call it the Skolemite's meta-theory (SMT). For the sake of the argument, we are willing to concede that there are only countable sets in SMT, but, on pain of self-refutation or unintelligibility, we also assume that SMT is a meta-theory where concepts are fixed and absolute. Among other things, in SMT we have a clear grasp of the axioms of ZFC; not only this is a standard assumption in every form of Skolem Paradox, but it is also essential to get the Skolemite's argument off the ground. Interestingly, Hartogs Theorem is a theory of ZFC.

The key observation for our response is that if the Skolemites are willing to iterated the relativity criticism moving from theory to meta-theory, then they will end up iterating their response for every countable ordinal; where here "every countable ordinal" should be understood in SMT. But in doing so, they will have produced an uncountable collection, namely ω_1; or else, the ω^+ they have in SMT. Of course, the Skolemite will object that the domain of their iteration function does not form a set, since it is uncountable, but still, the fact that this collection is uncountable will show that their meta-theory is uncountable. Another way to put it is that the Skolemite's $\mathbf{V}$ is uncountable. Notice that $\mathbf{V}$, being $\mathbf{V}$, cannot be looked from outside. It is all there is; for the Skolemite. There is no way for the Skolemite to go beyond their $\mathbf{V}$. Indeed, the uncountability of ω^+ is obtained within SMT and therefore cannot be accused of relativity, unless SMT is also relative. But if this is the case, then the Skolemite's argument looses all its force.[11] It is like if the Sceptic was accused of scepticism, thus, destroying the ground on which any sceptical conclusion can be drawn from a sceptical argument. Therefore, in order for the Skolemites not to taste their own medicine, they need to accept that their meta-theory contains an absolute notion of uncountability. This is why the iterated objection is self-refuting.

[11]All the insistence on the relativity of the Skolemite set-theoretical concepts that we find in this argument is something that we can point at, when discussing the Skolemite position, but it is not something that the Skolemite can do, since they live in their meta-theory. Of course from the perspective of someone who believe in the existence of uncountable collections, we can only hope that SMT is the actual meta-theory; the one we call $\mathbf{V}$. If not, too bad for the Skolmites: their argument is even less meaningful. This would only show the harmless relativity of model-theory, which is not an issue for the defender of the uncountable.

As in the previous response, we did not show that there are uncountable sets in our models, but only that the meta-theory in which the Skolemite argument is developed contains uncountable collections. But this is enough to counter the argument and to leave open the task of set theory to describe un uncountable universe of sets: the paradise that Cantor created for us.

4.3. **Answer to the fourth block.** We finally came to discuss Countabilism: the position according to which set theory should really be about countable collections. There is a jump here. So far, the Skolemites suggested the relativity of the notion of uncountability and the non existence of uncountable collections. This line of argumentation was based on the acceptance of ZFC and the use of LST. Yet, in this last bit of the discussion a new set theory is proposed: one where everything is countable and that is based on different principles (and logic). Therefore, Countabilism is not exactly in continuity with Skolem Paradox, since it undermines the premises that led the Skolemite to its proposal. This is why Countabilism rests on a somewhat different motivation.

The motivation for Countabilism comes from a modal perspective on set theory, where the notion of existence is replaced by that of possible existence. In this context, principles like Naïve Comprehension, normally responsible for paradoxes like Russell's, are substituted with their modal versions, where the actual existence of paradoxical sets is replaced by their possible existence. In this way paradoxes are defuses with modal tools and a new potentialist picture of set theory emerges [17–19]. In a similar vain, CaT can be seen as a result about the non-existence of bijections between, say, $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ and the same modal treatment can be applied to the way these bijections (do not) exist. Indeed, as long as we have means to extend the models of set theory, the non existence of a bijections (witnessing countability) can be understood only as relative to a model of ZFC and therefore not in contradiction with its possible existence in a larger models. In other terms, even if we do not have bijections, with $\mathbb{N}$, for every set in a model of ZFC, these bijection *can* exist in larger models. The external perspective on a model put forward by the use of LST is here substituted, symmetrically, by the internal perspective of model expansion.

The modality of this approach to mathematical existence can be easily modelled in the context of Kripke frames, where the possible worlds satisfy ZFC (i.e. they are models of ZFC) and the accessibility relation keeps track of the relation of model-extension, responsible for the existence of new bijections in larger models. It is here that the analogy between Countabilism and the previous discussion on the iterated argument of the Skolemite becomes apparent. If the previous block of the Skolmite's argument consisted of an iteration of the collapse of theory and meta-theory, Countabilism now deals with what happens at the limit of this iteration, when, so to say, everything is collapsed to countable. The picture offered by Countabilism is that of successive extensions that gradually swallow up every uncountable cardinal, thus realising all possibilities of being countable.

In order to assess Countabilism, first we shall describe the ways to approach this view and then we will discuss the arguments in favour of it; and how to respond to them. A quick look at the literature shows that

Countabilism is mostly presented only as a theoretical possibility. In this form it should not appear in the Skolemite's argument, since the consistency of Countabilism does not pose any threat to the uncountable. Exactly as the existence of a countable model of ZFC is not problematic for those who defend a set theory populated of uncountable collections. Indeed, models are only partial perspectives on the universe of sets and the task of the set-theoretical work is exactly that of separating the wheat from the chaff.

4.3.1. *Two approaches to Countabilism.* We can motivate Countabilism in two ways: the model-theoretic approach and the axiomatic one. Let us analyse them in turn and assess their merits.

The former is closer to our discussion on Skolem Paradox, since it starts from a collections of models of ZFC, a multiverse, and by actualising possibilities (i.e. by moving along the accessibility relation of the multiverse viewed as a Kripke frame) collapses uncountable cardinals to ω, using forcing as the main techniques. So far so good, but in order to motivate Countabilism this approach needs not to depend on the notion of uncountable. Is this the case? More specifically, this question amounts to asking where this collapsing argument is performed. This question clearly echos the one we asked for the iterated argument of the Skolemite. And as before it is important to keep theory and meta-theory separated and to check whether we are not accidentally introducing uncountable collections in the meta-theory. This would be problematic, since it would be odd to need the notion of uncountable to motivate a position according to which everything is countable.

Unfortunately, the models proposed by Pruss in [21] and by Scambler in [23] not only uses uncountable cardinals, but even large cardinals; specifically, worldly cardinals in on case and Mahlo in the other. Both Pruss and Scambler are aware of the issue. Pruss, for example, writes "The thesis of this paper is that, for all that we know, all sets have the same count as the natural numbers. But the argument for the coherence of this thesis makes use of large-cardinal-style assumptions. This seems deeply problematic." [21, p. 10]. His answer to this problem is twofold. First, he notices that what is assumed is not the existence of the corresponding large cardinal, but only its consistency. In fairness, this defence is quite weak since it should provide reasons to believe in the consistency of large cardinals, besides their existence. In the last decades set-theorists have provided many good reasons for the truth of large cardinals; they range from intrinsic justifications (like Reflection Principles and maximisation of expressive power) to extrinsic ones (like their success and linearity) [2, 14]. All these good arguments have suggested that the best reason for assuming the consistency of large cardinals is actually their existence. Of course we are not appealing to authority here, but to literature. We only notice that the strength of this argument rests on the justification of a view on large cardinals that is not sufficiently developed.

The second reason that Pruss provides for dismissing the issue of using large cardinals for justifying Countabilism rests on the fact that "we could have a countable transitive model of [...] ZFC plus the existence of a large cardinal, and a larger model in which the first model becomes countable". But this is nothing else than the iterated objection of the Skolemite, that we have already answered using Hartogs Theorem. Indeed, notice that also here we have the right to assume that the meta-theory in which Pruss's answer is proposed should be fixed and non-relative, again on pain of unintelligibility.

On the other hand Scambler considers any attempt to justify Countabilism using model-theory (and with this large cardinals) hopeless.

> The axiomatic presentation brings certain dialectical benefits. For example, as I just mentioned, in [15] Meadows develops a picture on which all sets are countable, with recourse to the generic multiverse over a countable model of set theory. But in so doing, he encounters a prima facie revenge problem [15, p19, note 9]: namely, that there seem to be uncountably many structures in the generic multiverse itself, thus apparently undermining the claim under consideration. [. . .] Similarly, approaches that proceed principally model-theoretically through the generic multiverse and related notions tend not to secure the countability of all things, plural and singular. For instance the universe of the countable set M on which the generic multiverse is based is a class from the point of view of its generic multi- verse, and is therefore never collapsed to countability by set forcing. If we think of the classes in M as plural, then effectively what this means is we receive no guarantee that really all things are countable, only that all sets are.[12]

This is why Scambler prefers the axiomatic approach. This consists in simply proposing an axiomatic system meant to formalise the Countabilist's thesis. Consequently, the use of uncountable collections in the consistency proof for the axioms for Countabilism should not count as a justification of the correctness of this approach, or the truth of these axioms, but only as an epistemological reassurance of the viability of this position. Scambler is clear on this point: "By the way, like each of Pruss and Meadows, I do not conclude from considerations such as these that all sets are really countable after all." [23, p. 1099]. But if the consistency of an axiom, or an axiomatic system in the case of Scambler, does not commit to truth, but only to the possibility of its truth, then Countabilism is not in conflict with the view that the universe of sets is uncountable. It is only another picture of sets that we can developed in a possibly uncountable universe of all sets.

So far we considered two approaches to Countabilism: one based on model theory and the other on the axiomatic method. We argued, following [23], that the former does not succeed in motivating this view, while the former can, keeping a neutral perspective on its truth. What remains to be done is to assess the justification of Countabilism, once its consistency has been provided.

4.3.2. *Arguments in favour of Countabilism.* As we have seen, many discussion of Countabilism are not aimed at its defence, but only at showing the consistency of this position. However, we find in the literature at least one place where Countabilism is explicitly defended: [8]. We are back in the

[12] [23, p. 1099].

Skolemite's argument and, then, our task is to critically assess it. From [8] we can extract two general comments and two arguments in favour of Countabilism. We will see that the two comments do not represent a serious threat to ZFC, while that the two arguments are only reformulations of Skolemite's arguments that we have already encountered.

We can start from the comments. The first one is of a conceptual nature: "Countabilism nicely corresponds to an intuitive understanding of infinity as limitless, such that it cannot itself be surpassed in number, and hence can only have one 'size'." [8, p. 2220]. The limitless of infinity seems here connected to a notion of indeterminacy and infinity seems to be out of reach because it is too big to be counted. This can be a point in favour of Countabilism as long as it is a feature that the Cantorian approach to cardinality lacks. Unfortunately, this is not the case. Indeed, the concept of absolute infinity (i.e. that of the universe of all sets, the class of all ordinals or cardinals) is a notion of infinity that does not admit of any specification and that has one size that surpasses any other; finite or transfinite. This idea of limitless, therefore, is well described also in the conception of infinity displayed by ZFC and it is treated syntactically in terms of definable classes. Therefore, we can argue that we do not loose any conceptual expressivity in approaching the notion of infinity in Cantorian terms; quite the contrary: we gain the possibility of a more nuanced treatment of infinity, even in the context of a limitless, unmeasurable notion of absolute infinity.

The second comment is historical. The authors in [8] claim that ZFC + Projective Determinacy (PD) provides a clear picture of $H(\aleph_1)$ (i.e. the collection of all hereditarily countable sets), but that we lack an analog result for $H(\aleph_2)$. This should count as a good indication that "Countabilism exactly predicts the current state of affairs regarding set-theoretic truth. It predicts that truths about (hereditarily) countable sets should be within reach, but truths about the uncountable should lapse into indeterminacy." [8, p. 2225]. This comment is highly problematic for two distinct reasons. First of all it is an empirical claim about the current state of research in set theory, which, therefore, lacks theoretical value and can be falsified in the future. The second reason is that, actually, *there are* axioms that are analog of PD for $H(\aleph_2)$, these are Forcing Axioms; especially now that they have been shown to provide the approach to $H(\aleph_2)$ that we obtain by assuming Woodin's axiom $(*)$ [1]. Not only they provide an empirical completeness of $H(\aleph_2)$, but they also rest on the same theoretical reason that explain and justify the of PD. Indeed, both PD and Forcing Axioms provide enough existential closure to enforce the right degree of model completeness to the relevant initial segments of the cumulative hierarchy [25, 26].

We can now analyse the arguments in favour of Countabilism. The first one is centred around set-theoretical independence and on the possibility to change the cardinality of a set by forcing. This argument is divided in two step. First forcing is used to argue in favour of width-potentialism and then the connection, again based on forcing, between width-potentialism and Countabilism is used to embrace the latter in view of the good reasons we have for the former. Let us see how the argument goes. The authors of [8] argue that "The absence of any explanation for why the maximal powerset of $\mathbb{N}$ should 'stop' at one cardinality rather than another constitutes an explanation for why there can be no maximal powerset." In turn, the absence of a maximal powerset for $\mathbb{N}$ is viewed as an argument in favour of width-potentialism, which is considered as an essential component of Countabilism. Strangely enough, this argument for width-potentialism seems to be argument against Countabilism. The reason being, that the possibility to blow up the cardinality of the Continuum to larger and larger values cannot reasonably count in favour of a universe where everything is countable. Quite the opposite. If we make $\mathcal{P}(\mathbb{N})$ a proper class, then we will have that the reals surpass in cardinality every cardinal, even the uncountable ones. The authors seem aware of this issue and in fact they add "It is still not obvious, however, how one gets an explanation of Countabilism from an explanation of width-potentialism. One way one can bridge the divide is by appealing to the set-theoretic technique of forcing. [...] forcing vindicates Countabilism insofar as uncountable cardinalities can always be seen to be countable in appropriate forcing extensions." But this is nothing else than Skolem Paradox or the iterated version of the Skolemite's argument, that can easily be addressed using CoT or HartogsTheorem and noticing, again, that the meta-theory where this argument is performed needs to remain fix, on pain of unintelligibility of what it means to be uncountable in a larger model. Moreover, this argument goes back to the model-theoretic approach to countabilism that is not favoured even from their proponents [23].

The last argument for Countabilism is based on the process of generation of ordinals. The authors of [8] argue that using Cantor's first two principles of generations (i.e. taking the successor of an ordinal or the limit of previously generated ordinals) as described in [10], we will never go beyond the realm of countable ordinals. This is true, but Cantor also presents a third principle of generation that is nothing else than the application of Hartogs' function to the collection of ordinals of a given cardinality, which produces the first ordinal of greater cardinality. In this case, the third principle applied to countable ordinals produces ω_1. As before, our answer to this argument rests on showing that the Skolemite's criticism needs to happen in a meta-theory where uncountable collections exist. In order to see this, we need to ask where this process of generation takes place. In other words, in what context are we considering the collection of all countable ordinals? Whatever this might be (the Skolemite's meta-theory), Hartogs Theorem tells us that it must be an uncountable context. Indeed, the meta-theory in which we can talk of any countable ordinals needs to be uncountable, in order to include all such ordinals. And again, this totality needs to be maximal for the Skolemite, on pain of unintelligibility of its own notion of countable ordinal. Again we find that the universe where Countabilism lives needs to be uncountable; thus falsifying its conceptual motivation.

Therefore, both arguments in favour can be easily answered as before and they do not add new elements to the Skolemite's criticism. Indeed, we showed that results like CoT and Hartogs Theorem suggest, contrary to Countabilism, that the universe of all set is uncountable. As before, we did not show that there are uncountable collections, but only that the meta-theory where arguments in favour of Countabilism are produced needs to be uncountable. And this is enough to put into question the conceptual premises of Countabilism.

5. Conclusion

In this paper we analysed Skolemite's arguments against the uncountable and we found that they rest on a confusion of levels between theory and meta-theory. By clearly separating the two, we showed that formal results, by Cantor, Cohen, and Hartogs, suggest that the universe of set (whatever this is) is uncountable. Interestingly, this applies to the meta-theory of the defender of the uncountable as well as to that of the Skolemite. Consequently, even if we can consistently entertain the idea that everything is countable, still, to be able to entertain this idea, we need the notion of a greater infinity. This is what the uncountable is: a size of infinity that exceed incommensurably the countable one. The strength of Cantor's theory of transfinite cardinalities consists in showing that such a notion admits specification and a deep and fascinating analysis. Then, why deprive ourselves of such a fruitful theory, when the only way to negate it is to share its premises: the existence of notion of infinity that goes beyond that of the natural numbers? Clearly there is no need, since, although consistent, the possibility that every set is countable is only one of the many different pictures of the universe that model theory provides us. As any countable model of ZFC shows, not all of its properties reflects faithfully how the universe of all sets is. This is the lesson we can learn from Skolem Paradox and that suggests that the study of set theory should consists of a delicate balance of intra-theoretical and meta-theoretical considerations. We leave to another occasion to describe how this equilibrium can be attained. For now we just content ourselves to have defended the value of this enterprise and, with it, the notion of uncountability in set theory.

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